Definition 4.8:

Let $a > 0$ and define the function $\exp_a : \mathbb{R} \to \mathbb{R}$ to be
$$\exp_a(x) := \exp(x \log a)$$

Proposition 4.7:

The function $\exp_a : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and we have:

i) $\exp_a(x+y) = \exp_a(x)\exp_a(y) \quad \forall \; x, y \in \mathbb{R}$

ii) $\exp_a(n) = a^n \quad \forall \; n \in \mathbb{Z}$.

iii) $\exp_a\left(\frac{p}{q}\right) = \sqrt[q]{a^p} \quad \forall \; p \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \geq 2$.

Proof:

$\exp_a$ is the composition of the two continuous functions $x \mapsto x \log a$ and $y \mapsto \exp(y)$

Prop. 4.3 $\Longrightarrow$ $\exp_a$ is continuous

i) We compute
$$\begin{aligned} \exp_a(x+y) &= \exp((x+y)\log a) \\ &= \exp(x \log a)\exp(y \log a) \\ &= \exp_a(x) \cdot \exp_a(y) \end{aligned}$$

We also get for $y = -x$: $\exp_a(-x) = \dfrac{1}{\exp_a(x)}$

ii) $\exp_a(nx) = \exp_a(x+(n-1)x) \overset{i)}{=} \exp_a(x)\exp_a((n-1)x)$

$$= \cdots = \underbrace{\exp_a(x) \cdot \cdots \cdot \exp_a(x)}_{n \text{ times}}$$

$$= (\exp_a(x))^n \quad \text{for } n \in \mathbb{N} \text{ and } x \in \mathbb{R}. \quad (*)$$

Furthermore, $\exp_a(1) = \exp(\log a) = a$ and $\exp_a(-1) = \frac{1}{a}$. Together with $(*)$ this gives for $x=1$ and $x=-1$:

$$\exp_a(n) = a^n \quad \text{and} \quad \exp_a(-n) = a^{-n}$$

iii) We compute

$$a^p = \exp_a(p) = \exp_a\left(q \cdot \frac{p}{q}\right) = \left(\exp_a\left(\frac{p}{q}\right)\right)^q$$

$$\Rightarrow \sqrt[q]{a^p} = \exp_a\left(\frac{p}{q}\right)$$

$\square$

<u>Corollary 4.3:</u>

For all $a > 0$ we have $\lim_{n\to\infty} \sqrt[n]{a} = 1$

<u>Proof:</u>

Using continuity of $\exp_a$ we get:

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \exp_a\left(\frac{1}{n}\right) = \exp_a(0) = 1.$$

$\square$

Definition 4.9:

Proposition 4.7 justifies the notation

$$a^x := \exp_a(x) = \exp(x \log a)$$

We call a the "base" of the exponential.
We also define the "euler number":

$$e := \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \dots$$

$$= 2.7182818\dots$$

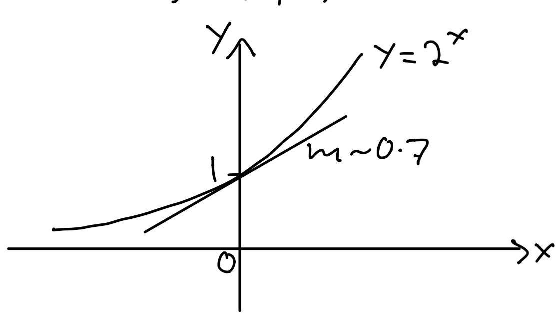
Remark 4.3:

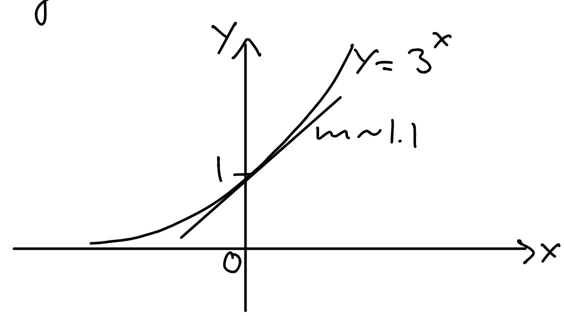
i) An equivalent definition of e is:

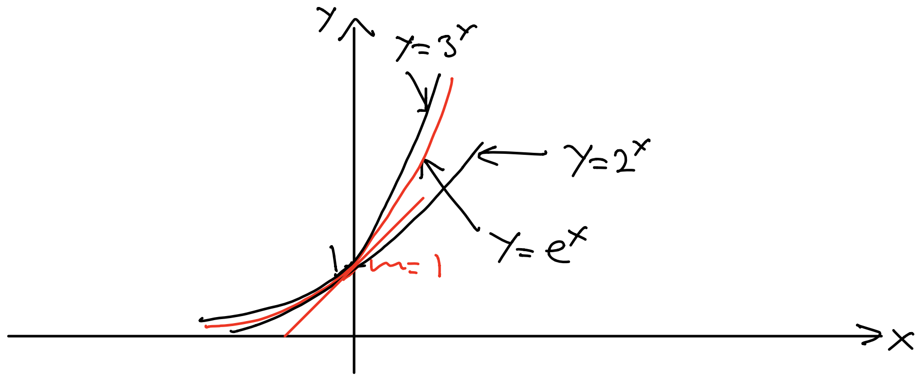
$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \quad \text{(see Homework 1)}$$

Later on we will show that this is equivalent to Definition 4.9.

ii) Of all possible bases for an exponential, the base e has the unique property that the slope of the tangent at the point $(x, y) = (0, 1)$ is exactly 1:







We will prove this property later when we introduce differentiation.

Proposition 4.8:

For all $a, b \in \mathbb{R}_{>0}$ and $x, y \in \mathbb{R}$ we have:

i) $a^x a^y = a^{x+y}$,

ii) $(a^x)^y = a^{xy}$,

iii) $a^x b^y = (ab)^x$,

iv) $(1/a)^x = a^{-x}$.

Proof:

i) Just a different way of writing Prop. 4.7 i)

ii) $a^x = \exp(x \log a) \Rightarrow \log(a^x) = x \log a$,

and therefore $(a^x)^y = \exp(y \log(a^x))$

$$= \exp(yx \log a) = a^{xy}$$

iii), iv) analogous $\square$

Proposition 4.9 (limits):

i) For all $K \in \mathbb{N}$ we have $\lim_{x \to \infty} \frac{e^x}{x^K} = \infty$

or in other words: "The exponential grows faster than any polynomial."

Proof:

For all $x > 0$ we have:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > \frac{x^{K+1}}{(K+1)!}$$

$\Rightarrow \frac{e^x}{x^K} > \frac{x}{(K+1)!}$ from which the claim follows immediately.

$\square$

ii) For all $K \in \mathbb{N}$:

$\lim_{x \to \infty} x^K e^{-x} = 0$ and $\lim_{x \to 0} x^K e^{\frac{1}{x}} = \infty$

Proof:

rewrite $x^K e^{-x} = \left(\frac{e^x}{x^K}\right)^{-1} \Rightarrow$ first statement

$$\lim_{x \to 0} x^K e^{\frac{1}{x}} = \lim_{y \to \infty} \left(\frac{1}{y}\right)^K e^y = \lim_{y \to \infty} \frac{e^y}{y^K} = \infty$$

$\square$

iii) $\lim_{x \to \infty} \log x = \infty$ and $\lim_{x \to 0} \log x = -\infty$

Proof:

Let $K \in \mathbb{R}$ be arbitrary. As log is strictly monotonic increasing, we have $\log x > K$ for all $x > e^K$. Thus $\lim_{x \to \infty} \log x = \infty$

Also
$$\lim_{x \to 0} \log x = \lim_{y \to \infty} \log(1/y) = -\lim_{y \to \infty} \log y = -\infty$$
$\square$

iv) For every real number $\alpha > 0$ we have
$$\lim_{x \to 0} x^\alpha = 0 \quad \text{and} \quad \lim_{x \to 0} x^{-\alpha} = \infty .$$

Proof:

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers with $x_n > 0$ and $\lim_{n \to \infty} x_n = 0$. iii) gives
$$\lim_{n \to \infty} \alpha \log x_n = -\infty$$
Together with $\lim_{y \to -\infty} e^y = 0$ (see ii)) we get
$$\lim_{n \to \infty} x_n^\alpha = \lim_{n \to \infty} e^{\alpha \log x_n} = 0,$$
and so $\lim_{x \to 0} x^\alpha = 0$. The second statement follows due to $x^{-\alpha} = \frac{1}{x^\alpha}$.

$\square$

v) For all $\alpha > 0$ we have $\lim_{x \to \infty} \frac{\log x}{x^\alpha} = 0$

or in other words: "The logarithm grows slower for $x \to \infty$ than any polynomial"

Proof:

Let (x_n) be a sequence of positive numbers satisfying $\lim x_n = \infty$. For the sequence

$$y_n := \alpha \log x_n$$

we have due to iii): $\lim_{n\to\infty} y_n = \infty$

As $x_n^\alpha = e^{y_n}$, we obtain (using ii)):

$$\lim_{n\to\infty} \frac{\log x_n}{x_n^\alpha} = \lim_{n\to\infty} \frac{1}{\alpha} y_n e^{-y_n} = 0$$

$\square$

vi) For all $\alpha > 0$ we have $\lim_{x\to 0} x^\alpha \log x = 0$

Proof:

This follows from v) as $x^\alpha \log x = -\frac{\log\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)^\alpha}$

$\square$